

## Lecture 3: Measurement and Quantum Teleportation

*“I think that a particle must have a separate reality independent of measurements. That is, an electron has spin, location and so forth even when it is not being measured. I like to think the moon is there even if I am not looking at it.”*

— Albert Einstein

*“... experiments have now shown that what bothered Einstein is not a debatable point but the observed behaviour of the real world.”*

— N. David Mermin

### 1 Postulate 4: Measurement

We have arrived at the final postulate of quantum mechanics, which asks: How does one mathematically model the act of *measuring* or *observing* a quantum system? For example, suppose we run a quantum algorithm to compute the prime factors of a large composite integer — how can we *measure* the answer register to obtain a classical output string which encodes the prime factors? At this point it may come as no surprise to you that here again quantum mechanics plays an unexpected card — it turns out that the very act of looking at or *observing* a quantum system irreversibly alters the state of the system! This is precisely the phenomenon Einstein’s quote above refers to — it’s like saying the “state” of the moon is only well-defined the moment you *look* at it.

To model this phenomenon, we shall use the notion of a *projective* or *von Neumann* measurement (named after Hungarian child prodigy and mathematician, John von Neumann, who was apparently already familiar with calculus at the age of 8). To do so, we must define three classes of linear operators, each of which is increasingly more restricted. All three classes will prove vital throughout this course.

#### Hermitian, positive semi-definite, and orthogonal projection operators.

1. *Hermitian operators*: An operator  $M \in \mathcal{L}(\mathbb{C}^d)$  is *Hermitian* if  $M = M^\dagger$ . Examples you are already familiar with are the Pauli  $X$ ,  $Y$ , and  $Z$  gates, which are not only unitary, but also Hermitian. As you have shown in Assignment 1, a Hermitian operator has the important property that all of its eigenvalues are *real*. Thus, Hermitian operators can be thought of as a higher dimensional generalization of the real numbers.

**Exercise.** Verify that Pauli  $Y$  is Hermitian. Is the Hadamard gate Hermitian?

2. *Positive semi-definite operators*: If a Hermitian operator has only *non-negative* eigenvalues, then it is called *positive-semidefinite*. Thus, positive semi-definite (or positive for short) matrices generalize the non-negative real numbers.

**Exercise.** Prove that Pauli  $X$  and  $Z$  are not positive semi-definite. (Hint: Recall the spectral decompositions of  $X$  and  $Z$ .)

3. *Orthogonal projection operators*: A Hermitian matrix  $\Pi \in \mathcal{L}(\mathbb{C}^d)$  is an *orthogonal projection operator* (or projector for short) if  $\Pi^2 = \Pi$ . This is equivalent to saying  $\Pi$  has only eigenvalues 0 and 1.

Let us prove this equivalence briefly: Since  $\Pi$  is Hermitian, we can take its spectral decomposition,  $\Pi = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ . Hence,

$$\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| = \Pi = \Pi^2 = \left( \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \right) \left( \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j| \right) = \sum_i \lambda_i^2 |\lambda_i\rangle\langle\lambda_i|,$$

where the last inequality follows since  $\{|\lambda_i\rangle\}$  is an orthonormal basis. Since the  $|\lambda_i\rangle$  are orthogonal, we thus have that for all  $i$ ,  $\lambda_i = \lambda_i^2$ . But this can only hold if  $\lambda_i \in \{0, 1\}$ , as claimed.

**Exercise.** Verify that  $I$ ,  $|0\rangle\langle 0|$ , and  $|1\rangle\langle 1|$  are all projectors. More generally, show that for arbitrary unit vector  $|\psi\rangle \in \mathbb{C}^d$ ,  $|\psi\rangle\langle\psi|$  is a projector.

It is important to note that since a projector  $\Pi$ 's eigenvalues are all 0 or 1, its spectral decomposition must take the form  $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ , where  $\{|\psi_i\rangle\}$  are an orthonormal set. Conversely, summing any set of orthonormal  $\{|\psi_i\rangle\}$  in this fashion yields a projector, as you will now show.

**Exercise.** Let  $\{|\psi_i\rangle\}$  be an orthonormal set. Prove that  $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$  is a projector.

Observe that a projector  $\Pi$  has rank 1 if and only if  $\Pi = |\psi\rangle\langle\psi|$  for some  $|\psi\rangle \in \mathbb{C}^d$ , since the rank of  $\Pi$  equals<sup>1</sup> the number of non-zero eigenvalues of  $\Pi$ , and here  $\Pi = |\psi\rangle\langle\psi|$  is a spectral decomposition of  $\Pi$ . Finally, let us develop an intuition for what a projector actually *does* — for any projector  $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$  and state  $|\phi\rangle$  to be measured, we have

$$\Pi|\phi\rangle = \left( \sum_i |\psi_i\rangle\langle\psi_i| \right) |\phi\rangle = \sum_i |\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_i \langle\psi_i|\phi\rangle |\psi_i\rangle \in \text{Span}(\{|\psi_i\rangle\}),$$

where note  $\langle\psi_i|\phi\rangle \in \mathbb{C}$ . Thus,  $\Pi$  projects us down onto the span of the vectors  $\{|\psi_i\rangle\}$ .

**Exercise.** Consider three-dimensional vector  $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \in \mathbb{C}^3$  and  $\Pi = |0\rangle\langle 0| + |1\rangle\langle 1|$ . Compute  $\Pi|\phi\rangle$ , and observe that the latter indeed lies in the two-dimensional space  $\text{Span}(\{|0\rangle, |1\rangle\})$ .

**Projective measurements.** With projectors in hand, we can now define a projective measurement. A *projective measurement* is a set of projectors  $B = \{\Pi_i\}_{i=0}^m$  such that  $\sum_{i=0}^m \Pi_i = I$ . The latter condition is called the *completeness* relation. If each  $\Pi_i$  is rank one, i.e.  $\Pi_i = |\psi_i\rangle\langle\psi_i|$ , then we say that  $B$  models a *measurement in basis*  $\{|\psi_i\rangle\}$ . Often, we shall measure in the computational basis, which is specified by  $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$  in the case of  $\mathbb{C}^2$  (and generalizes as  $B = \{|i\rangle\langle i|\}_{i=0}^{d-1}$  for  $\mathbb{C}^d$ ).

**Exercise.** Verify that  $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$  is a projective measurement on  $\mathbb{C}^2$ .

With a projective measurement  $B = \{\Pi_i\}_{i=0}^m \subseteq \mathbb{C}^d$  in hand, let us specify how one *uses*  $B$ . Suppose our quantum system is in state  $|\psi\rangle \in \mathbb{C}^d$ . Then, the probability of obtaining outcome  $i \in \{0, \dots, m\}$  when measuring  $|\psi\rangle$  with  $B$  is given by

$$\Pr(\text{outcome } i) = \text{Tr}(\Pi_i |\psi\rangle\langle\psi| \Pi_i) = \text{Tr}(\Pi_i^2 |\psi\rangle\langle\psi|) = \text{Tr}(\Pi_i |\psi\rangle\langle\psi|),$$

where the second equality follows by the cyclic property of the trace and the third since  $\Pi_i$  is a projector.

<sup>1</sup>This holds since  $\Pi$  is Hermitian, and hence also normal.

**Exercise.** Let  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$ . Show that if we measure in the computational basis, i.e. using  $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ , then the probabilities of obtaining outcomes 0 and 1 are  $|\alpha|^2$  and  $|\beta|^2$ , respectively.

The exercise above has an important moral — requiring a quantum state  $|\psi\rangle$  to be a *unit* vector (i.e.  $|\alpha|^2 + |\beta|^2 = 1$ ) ensures that when measuring  $|\psi\rangle$ , the distribution over the outcomes is a valid *probability distribution*, i.e. the probabilities for all possible outcomes sum to 1. The other important take-home message here is that measurements in quantum mechanics are inherently *probabilistic* — in general, the outcomes cannot be perfectly predicted!

Finally, we started this lecture by saying that the very act of measuring a quantum state disturbs the system. Let us now formalize this; we will crucially use the fact discussed earlier that a projector *projects* a vector  $|\psi\rangle$  down into a smaller subspace. Specifically, upon obtaining outcome  $\Pi_i$  when measuring  $B$ , the state of system “collapses” to

$$\frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\text{Tr}(\Pi_i|\psi\rangle\langle\psi|\Pi_i)} = \frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\text{Tr}(\Pi_i|\psi\rangle\langle\psi|)}. \quad (1)$$

Note the denominator above is a scalar, and is just the probability of outcome  $i$ . There are two points here which may confuse you: Why have we written the output state as a matrix  $\Pi_i|\psi\rangle\langle\psi|\Pi_i$  rather than a vector  $\Pi_i|\psi\rangle$ , and what is the role of the denominator? Let us handle each of these in order.

First, conditioned on outcome  $\Pi_i$ , the output state is indeed a vector, namely  $\Pi_i|\psi\rangle$ . However, there is a more general formalism which we shall discuss shortly called the *density operator formalism*, in which quantum states are written as matrices, not vectors. Specifically, the “density matrix” representing vector  $|\psi\rangle$  would be written as matrix  $|\psi\rangle\langle\psi|$ . The density operator formalism is more general than the state vector approach we have taken so far, and will be crucial for studying individual subsystems of a larger composite quantum state. Thus, the answer to question 1 is that we have written the output as a matrix simply to slowly ease the transition into the density matrix formalism.

The motive behind question 2 is somewhat less sinister — the problem here is that since we projected out part of  $|\psi\rangle$  during the measurement, the output  $\Pi_i|\psi\rangle$  may not necessarily be normalized. To renormalize  $\Pi_i|\psi\rangle$ , we simply divide by its Euclidean norm to obtain

$$|\psi'\rangle = \frac{\Pi_i|\psi\rangle}{\|\Pi_i|\psi\rangle\|_2} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i\Pi_i|\psi\rangle}} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}}.$$

The state  $|\psi'\rangle$  describes the post-measurement state of our system, assuming we have obtained outcome  $i$ .

**Exercise.** Show that (the density matrix)  $|\psi'\rangle\langle\psi'|$  equals the expression in Equation (1).

**Exercise.** Let  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$ . Show that if we measure in the computational basis, i.e. using  $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ , and obtain outcome  $i \in \{0, 1\}$ , then the post-measurement state is  $|i\rangle$  (or  $|i\rangle\langle i|$  in density matrix form).

A final quirk we should iron out is the following — in terms of measurements, what is the consequence of the fact that a projector  $\Pi_i$  satisfies  $\Pi_i^2 = \Pi_i$ ? Well, if you observe a quantum system now, and then again five minutes from now, and if the system has not been subjected to any gates or noise in between the measurements, then the two measurement results you obtain should agree (I realize the study of quantum mechanics has likely distorted your view of when you can trust your intuition, but this is one case in which you can). To model this, suppose we measure using  $B = \{\Pi_i\}$  and obtain results  $i$  and  $j$  in measurements 1 and 2, respectively. Then:

$$\text{Pr}(\text{outcome } j \mid \text{outcome } i) = \text{Tr}\left(\Pi_j \frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\text{Tr}(\Pi_i|\psi\rangle\langle\psi|)} \Pi_j\right) = \frac{\text{Tr}(\Pi_j\Pi_i|\psi\rangle\langle\psi|\Pi_i\Pi_j)}{\text{Tr}(\Pi_i|\psi\rangle\langle\psi|)} \quad (2)$$

To simplify this expression, we use the fact that if the completeness relation holds for projectors  $\{\Pi_i\}$ , i.e.  $\sum_i \Pi_i = I$ , then it turns out that  $\Pi_j\Pi_i = \delta_{ij}\Pi_i$ , where recall  $\delta_{ij}$  is the Kronecker delta. Thus, if  $i \neq j$ ,

Equation (2) equals 0, and if  $i = j$ , it reduces to

$$\frac{\text{Tr}(\Pi_j \Pi_i |\psi\rangle\langle\psi| \Pi_i \Pi_j)}{\text{Tr}(\Pi_i |\psi\rangle\langle\psi|)} = \frac{\text{Tr}(\Pi_i^2 |\psi\rangle\langle\psi| \Pi_i^2)}{\text{Tr}(\Pi_i |\psi\rangle\langle\psi|)} = \frac{\text{Tr}(\Pi_i |\psi\rangle\langle\psi| \Pi_i)}{\text{Tr}(\Pi_i |\psi\rangle\langle\psi|)} = \frac{\text{Tr}(\Pi_i |\psi\rangle\langle\psi|)}{\text{Tr}(\Pi_i |\psi\rangle\langle\psi|)} = 1,$$

i.e. measuring the state a second time again yields outcome  $i$  with probability 1, as desired. Thus, although observing a state for the first time disturbs it, subsequent measurements will consistently return the same measurement result!

**Exercise.** Suppose we measure  $|0\rangle$  in basis  $B = \{|+\rangle\langle+|, |-\rangle\langle-|\}$ . What are the probabilities of outcomes  $+$  and  $-$ , respectively? What are the post-measurement states if one obtains outcome  $+$  or  $-$ , respectively?

We close this section by giving the circuit symbol which denotes a measurement of a qubit  $|\psi\rangle \in \mathbb{C}^2$  in the computational basis  $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ :

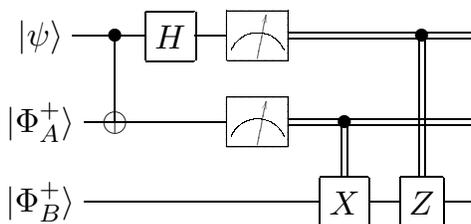


The double-wires on the right side indicate that the output of the measurement is a classical string (indicating which measurement outcome was obtained).

## 2 Quantum teleportation

With the concepts of the Bell state and measurement in hand, we can discuss our first neat computational trick: Quantum teleportation. Suppose you have a single-qubit quantum state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  in your possession (i.e. as a physical system, not on paper), but that you do not know the values of  $\alpha$  and  $\beta$ . Your friend Alice now phones you and asks to borrow your state. How can you send it to her? One obvious way is simply to pop your system in the mail and physically send it over. However, it turns out that by exploiting the phenomenon of entanglement, you can do something incredible — by sending two classical bits over the telephone to Alice, you can “teleport”  $|\psi\rangle$  instantly to her!

To teleport  $|\psi\rangle$ , we assume that you and Alice each share half of a Bell state  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  to begin with; specifically, you hold qubit 1 of  $|\Phi^+\rangle$ , and Alice holds qubit 2. The teleportation circuit is then given as follows:



Let us break this down piece by piece. The first two wires are held by you; wire 1 holds the state to be teleported,  $|\psi\rangle$ , and wire 2 holds your half of  $|\Phi^+\rangle$ . The third wire holds Alice’s half of  $|\Phi^+\rangle$ . Note that we have used  $|\Phi_A^+\rangle$  and  $|\Phi_B^+\rangle$  to denote the two “halves” of  $|\Phi^+\rangle$ , but this is poor notation — read literally, this diagram suggests  $|\Phi^+\rangle = |\Phi_A^+\rangle \otimes |\Phi_B^+\rangle$ , which is *not* true since  $|\Phi^+\rangle$  is entangled, and hence from last lecture we know that there do not exist states  $|\Phi_A^+\rangle, |\Phi_B^+\rangle$  such that  $|\Phi^+\rangle = |\Phi_A^+\rangle \otimes |\Phi_B^+\rangle$ . This notation is chosen simply because I was unable to find a way to correctly display  $|\Phi^+\rangle$  across wires 2 and 3 in the brief time I dedicated to the task.

The circuit can be divided into 5 steps: Step 1 performs the CNOT, Step 2 the Hadamard gate, Step 3 measures qubits 1 and 2, Step 4 applies a conditional  $X$  gate, and Step 5 applies a conditional  $Z$  gate. The latter two require clarification: The conditional  $X$  gate here takes a classical bit  $b$  as input (hence the

incoming wire at the top is a double line), and applies  $X$  if and only if  $b = 1$ . The conditional  $Z$  gate is defined analogously.

Now that we have technically parsed this diagram, let us intuitively parse it. First, you begin in Steps 1 and 2 by performing a CNOT and Hadamard on your qubits, followed by a standard basis measurement in Step 3. Since a measurement in the standard basis maps each qubit to either  $|0\rangle$  or  $|1\rangle$ , the output of your two measurements can jointly be thought of as one of the four bit strings 00, 01, 10, or 11. Call these bits  $b_0b_1$ . Now you pick up the telephone, call Alice, and tell her the value of  $b_0b_1$ . Conditioned on  $b_0$ , she applies  $X$  to her half of the Bell pair, followed by  $Z$  conditioned on  $b_1$ . The claim is that at this point, Alice's qubit's state has been magically converted to  $|\psi\rangle$ . In fact, as we shall see shortly,  $|\psi\rangle$  has also disappeared from your possession! In this sense, teleportation has taken place.

Let us formally analyze the action of this circuit. Denote by  $|\psi_i\rangle$  for  $i \in \{0, \dots, 5\}$  the joint state of your and Alice's systems immediately after Step  $i$  has taken place. Here, we define  $|\psi_0\rangle$  as the initial joint state before any gates are applied; it is given by

$$|\psi_0\rangle = |\psi\rangle|\Phi^+\rangle = (\alpha|0\rangle + \beta|1\rangle) \left( \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \right) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle).$$

After Step 1, i.e. after the CNOT, we have state

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle).$$

After Step 2, i.e. after the Hadamard gate, we have

$$\begin{aligned} & \frac{1}{\sqrt{2}}(\alpha|+\rangle|00\rangle + \alpha|+\rangle|11\rangle + \beta|-\rangle|10\rangle + \beta|-\rangle|01\rangle) \\ &= \frac{1}{2}(\alpha(|0\rangle + |1\rangle)|00\rangle + \alpha(|0\rangle + |1\rangle)|11\rangle + \beta(|0\rangle - |1\rangle)|10\rangle + \beta(|0\rangle - |1\rangle)|01\rangle) \\ &= \frac{1}{2}(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)). \end{aligned} \quad (3)$$

Let us now pause and analyze the state of affairs. There are four terms in this superposition, each of which begins with a distinct bit string  $|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$ , or  $|11\rangle$ . This means that if you now measure qubits 1 and 2 in the standard basis and obtain outcome (say)  $|00\rangle$ , then Alice's qubit on wire 3 collapses to the only consistent possibility,  $\alpha|0\rangle + \beta|1\rangle$ . In this case, teleportation has already taken place.

**Exercise.** Let  $|\phi\rangle \in (\mathbb{C}^2)^{\otimes 3}$  denote the state in Equation (3). Suppose you now measure qubits 1 and 2 in the standard basis. This can be modelled by projective measurement

$$B = \{|00\rangle\langle 00| \otimes I, |01\rangle\langle 01| \otimes I, |10\rangle\langle 10| \otimes I, |11\rangle\langle 11| \otimes I\},$$

where we have  $I$  on qubit 3 since we are not measuring it. Show that the probability of outcome 00 is  $1/4$ . Next, show that conditioned on outcome 00, the post-measurement state collapses to  $|00\rangle(\alpha|0\rangle + \beta|1\rangle)$ .

More generally, the four possible outcomes upon measuring qubits 1 and 2 result in four distinct residual states on Alice's qubit as follows:

$$00 \mapsto \alpha|0\rangle + \beta|1\rangle \quad 01 \mapsto \alpha|1\rangle + \beta|0\rangle \quad 10 \mapsto \alpha|0\rangle - \beta|1\rangle \quad 11 \mapsto \alpha|1\rangle - \beta|0\rangle.$$

Thus, if you simply send the two bits  $b_0b_1$  encoding the measurement outcome to Alice, then regardless of the value of  $b_0b_1$ , she can recover your original state  $\alpha|0\rangle + \beta|1\rangle$  via the following identities:

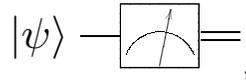
$$X(\alpha|1\rangle + \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle \quad Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle \quad ZX(\alpha|1\rangle - \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle.$$

**Exercise.** Verify that indeed  $ZX(\alpha|1\rangle - \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle$ .

In other words, by conditionally applying  $X$  and  $Z$  based on the outputs  $b_0b_1$  from your measurement, Alice can successfully recover your state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ . This is precisely what is depicted in Steps 4 and 5 of the teleportation circuit. Finally, note that since measuring your qubits leaves you in one of the four standard basis states  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ , the state  $|\psi\rangle$  has now “disappeared” from your possession!

### 3 Simulating arbitrary measurements via standard basis measurements

Let us continue to digest the quantum teleportation circuit with an observation. You may have noticed that in Section 1, we only described the circuit diagram representation of a *standard basis measurement* on a qubit, i.e.



whereas more generally projective measurements allow measuring in an *arbitrary* basis  $B = \{|\psi_1\rangle, |\psi_2\rangle\} \subseteq \mathbb{C}^2$ . It turns out that without loss of generality, we can restrict ourselves to measurements in the standard basis as follows. For this, we require an important fact about unitary matrices, which intuitively says that a unitary matrices maps orthonormal bases to orthonormal bases:

**Lemma 1.** Let  $\mathcal{B}_1 = \{|\psi_i\rangle\}_{i=1}^d \subseteq \mathbb{C}^d$  be an orthonormal basis. Then,  $\mathcal{B}_2 = \{U|\psi_i\rangle\}_{i=1}^d \subseteq \mathbb{C}^d$  is an orthonormal basis.

*Proof.* Since  $U$  is unitary, we have  $U^\dagger U = I$ . Therefore, for any  $i, j \in \{1, \dots, d\}$

$$(\langle\psi_i|U^\dagger)(U|\psi_j\rangle) = \langle\psi_i|\psi_j\rangle = \delta_{ij},$$

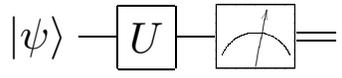
i.e.  $\mathcal{B}_2 = \{U|\psi_i\rangle\}_{i=1}^d$  is an orthonormal basis. □

In particular, for *any* orthonormal bases  $\mathcal{B}_1 = \{|\psi_i\rangle\}_{i=1}^d \subseteq \mathbb{C}^d$  and  $\mathcal{B}_2 = \{|\phi_i\rangle\}_{i=1}^d \subseteq \mathbb{C}^d$ , there exists a unitary  $U$  mapping  $\mathcal{B}_1$  to  $\mathcal{B}_2$ . This unitary is given by

$$U = \sum_{i=1}^d |\phi_i\rangle\langle\psi_i|.$$

**Exercise.** Verify that for any  $i \in \{1, \dots, d\}$ ,  $U|\psi_i\rangle = |\phi_i\rangle$ . Verify that  $U$  is unitary, i.e. that  $U^\dagger U = I$ . (Hint: Use the fact that for any orthonormal basis  $\{|\phi_i\rangle\}_{i=1}^d$ , we have  $\sum_i |\phi_i\rangle\langle\phi_i| = I$ .)

Let us now return to simulating a measurement in an arbitrary basis  $B = \{|\psi_1\rangle, |\psi_2\rangle\}$  on  $\mathbb{C}^2$  with standard basis measurements. By the exercise above, there exists a unitary  $U$  mapping  $B$  to the standard basis, i.e.  $U|\psi_1\rangle = |0\rangle$  and  $U|\psi_2\rangle = |1\rangle$ . Then, instead of measuring a state  $|\psi\rangle \in \mathbb{C}^2$  in basis  $B$ , one can equivalently measure  $U|\psi\rangle$  in the standard basis. In other words, we can simulate a measurement in  $B$  via the circuit diagram



To see why this works, consider the probability of obtaining outcome  $|\psi_1\rangle$  when measuring in  $B$ , which is

$$\Pr(\text{outcome } |\psi_1\rangle) = \text{Tr}(|\psi_1\rangle\langle\psi_1| \cdot |\psi\rangle\langle\psi|) = \text{Tr}((U^\dagger|0\rangle\langle 0|U)|\psi\rangle\langle\psi|) = \text{Tr}(|0\rangle\langle 0|(U|\psi\rangle\langle\psi|U^\dagger)).$$

In other words, one might as well measure state  $U|\psi\rangle$  against operator  $|0\rangle\langle 0|$ .

**Exercise.** Give a quantum circuit for measuring a qubit in the  $\{|+\rangle, |-\rangle\}$  basis.

Returning to the teleportation circuit, let us now observe that Steps 1 and 2 together are simply the *reverse* of our circuit from Lecture 2 which mapped the standard basis to the Bell basis. In other words, Steps 1 and 2 in teleportation map the Bell basis back to the standard basis — thus, Steps 1, 2, and 3 are simply measuring the first two qubits in the Bell basis, i.e. by projectors  $B = \{|\Phi^+\rangle\langle\Phi^+|, |\Psi^+\rangle\langle\Psi^+|, |\Phi^-\rangle\langle\Phi^-|, |\Psi^-\rangle\langle\Psi^-|\}$ .

## 4 Revisiting Schrödinger’s cat

In Lecture 2, we introduced Schrödinger’s cat, which through the magic of superposition, managed to be both alive and dead simultaneously, *so long as* the box containing the cat remained sealed. This begged the question: What happens if one opens the box and looks inside? Surely, one does not expect to find both a living and dead cat. The paradox here can finally be resolved via the measurement postulate — the act of opening the box and looking at the cat is itself a *measurement*. This measurement effectively collapses the superposition of dead and alive, leaving simply a cat which is either dead or alive. Note here that the very act of looking at something implies a measurement — this is because in order to “observe” the cat, photons must interact with the cat, and subsequently reach your eyes. This interaction and subsequent processing of the information contained in the photons by your brain “constitutes” a measurement.